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Relationships between different types of reflection equations and their Yang–Baxterization

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Abstract. Relationships between three different types of parameter-independent reflection equations, between their Yang–Baxterization and between the quantum group co-module structure of corresponding reflection algebras, are investigated.

Reflection equations (RE) were first introduced for the description of factorized scattering on a halfline [1], and applied recently to quantum current algebras [2] and integrable models with non-periodic boundary conditions [3, 4]. Kulish *et al* [5] proposed the concept of reflection algebras related to RE, studied their properties (quantum group co-module property) and constructed constant solutions of RE (i.e. one-dimensional representations of RA) [6]. The authors of this paper formulated an algebraic structure related to the eight-vertex model [7], studied representations of reflection algebras associated with $GL(n)_q$ [8–10] and established the Yang–Baxterization scheme of reflection equations for $\check{R}$ with square relations [11].

In different physical models the reflection equations take different forms. The following are the three most-used reflection equations:

$$\text{REI}: RK_1\tilde{R}K_2=K_2RK_1\tilde{R},$$

$$\text{REII}: RM_1R^{-1}M_2=M_2\tilde{R}^{-1}M_1\tilde{R},$$

$$\text{REIII}: RN_1R^{t_1}N_2=N_2R^{t_1}N_1R,$$

where $X_1=K\otimes I$, $X_2=I\otimes K$ $(X=K, M, N)$, $\tilde{R}=PRP$, $\tilde{R}^{-1}=PR^{-1}P$, P is the permutation operator $P(x\otimes y)=y\otimes x$, and t_1 denotes transposition w.r.t. the first space. In this letter we investigate the relationship between these three reflection equations, and their Yang–Baxterization. The relationship between the quantum group co-module structure of reflection algebras associated with these reflection equations is also revealed.

First, we discuss the relationship between REI and REII. In terms of $\check{R}=PR$ instead of R, REI and REII can be rewritten as

$$\check{R}K_1\check{R}K_1=K_1\check{R}K_1\check{R} \tag{1}$$

$$\check{R}M_1\check{R}^{-1}M_1=M_1\check{R}^{-1}M_1\check{R}. \tag{2}$$

Multiplying both sides of REI by $\check{R}^{-1}$ and then taking the inverse we have

$$\check{R}K_1^{-1}\check{R}^{-1}K_1^{-1}=K_1^{-1}\check{R}^{-1}K_1^{-1}\check{R}. \tag{3}$$

From (3) we conclude that, if K is inversive, then REI is equivalent to REII and $M=K^{-1}$. Here we would like to note that only inversive solutions to RE are of interest, though there exist solutions with vanishing solutions to RE.

We then turn to the relationship between REII and REIII. In this case we have to set up a relationship between R^{t_1} and R^{-1}. To this end, we suppose that $\check{R}$ satisfies the *crossing symmetry* [12, 13], namely, there exists a matrix C such that

$$R^{t_1}=C_1R^{-1}C_1^{-1} \tag{4}$$

where $C_1=C\otimes 1$ and $C^{-1}=\varepsilon C(\varepsilon\in\mathbb{C})$. The R-matrices, for example, associated with the fundamental representations of quantum algebras $so_q(n)$, $sp_q(n)$ [12] and with high-dimensional representations of $sl_q(2)$ [6], satisfy such a condition. It is easy to check from (4) that

$$[PC_1C_2, R]=[PC_1C_2, R^{-1}]=0. \tag{5}$$

Under the crossing symmetry (4), the REIII becomes

$$\check{R}N_1C_1\check{R}^{-1}N_1C_1=N_1C_1\check{R}^{-1}N_1C_1\check{R}. \tag{6}$$

Therefore REIII is equivalent to REII if R satisfies the crossing symmetry and $M=NC$. This conclusion enables us to obtain solutions to REII from those to REIII or vice versa. In summary, we have:

Proposition 1. If R satisfies crossing symmetry and K is inversive then REI, II and III are equivalent to each other and their solutions are connected by

$$K=M^{-1}=C^{-1}N^{-1}. \tag{7}$$

Let us see an example. For the R-matrix related to the fundamental representation of $sl_q(2)$

$$R=\begin{pmatrix} q & & & \\ & 1 & & \\ & \omega & 1 & \\ & & & q \end{pmatrix} \qquad \omega=q-q^{-1}. \tag{8}$$

the reflection algebra $\mathscr{A}_{\mathrm{III}}(R)$ related to REIII is generated by the four elements of the N-matrix

$$N=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{9}$$

which is subject to the following relations

$$\begin{aligned} &\alpha\gamma=q^2\gamma\alpha, \qquad [\beta,\gamma]=0, \qquad \gamma\delta=q^2\delta\gamma, \\ &[\alpha,\beta]=\omega\alpha\gamma, \qquad [\alpha,\delta]=\omega(q\beta\gamma+\gamma^2), \qquad [\beta,\delta]=\omega\gamma\delta. \end{aligned} \tag{10}$$

The N matrix has the inverse

$$N^{-1}=(\det{}_q N)^{-1}\begin{pmatrix} \delta & -\beta+\omega\gamma \\ -q^2\gamma & \alpha \end{pmatrix}$$

where the quantum determinant $\det_q N$ is the central element, and the metric C is

$$C=\begin{pmatrix} & 1 \\ -q & \end{pmatrix} \qquad C^{-1}=-q^{-1}C.$$

Therefore the reflection algebras $\mathscr{A}_{\mathrm{I}}(R)$ and $\mathscr{A}_{\mathrm{II}}(R)$ can be obtained by relation (7). They are

$$M=\begin{pmatrix} -q\beta & \alpha \\ -q\delta & \gamma \end{pmatrix} \qquad K=(\det_q N)^{-1}\begin{pmatrix} q\gamma & -q^{-1}\alpha \\ \delta & -\beta+\omega\gamma \end{pmatrix}. \tag{11}$$

These three quadratic algebras are obviously isomorphic.

It is well known that for the R given in (8), the metric C is a constant solution to REIII. In fact, it is true for any R satisfying the crossing symmetry.

Proposition 2. If R satisfies the crossing symmetry (4), then the metric C is a constant solution to REIII.

Then from proposition 1, we find that the identity matrix I (up to a constant) is the constant solution of REI and REII. This fact is obvious.

Finally, we give an exotic example, a special eight-vertex model

$$R=\begin{pmatrix} 1 & & & 1 \\ & \omega & 1 & \\ & 1 & \omega & \\ 1 & & & 1 \end{pmatrix}$$

where $\omega^2=1$. In this case we have

$$R^{t_1}=R.$$

Then by making use of the fact $[R, P]=0$, we can prove that REIII is equivalent to REI and $K=N$.

As is known, reflection algebras have an important property, namely, they are the quantum group co-module algebras. For a given R, we denote the reflection algebra by $\mathscr{A}_i(R)$ ($i=\mathrm{I,II,III}$) and the quantum group by $A(R)$. Accurately speaking, $A(R)$ should be the quantum matrix algebra of quantum group which is generated by the unit 1 and the elements of the T-matrix subject to the relations

$$RT_1T_2=T_2T_1R \tag{12}$$

where $T_1=T\otimes 1$, $T_2=1\otimes T$. Saying that $\mathscr{A}_i(R)$ is the $A(R)$ co-module algebra implies that there exists an algebra homomorphism $\varphi_i:\mathscr{A}_i(R)\rightarrow A(R)\otimes\mathscr{A}_i(R)$ such that

$$(\Delta\otimes\mathrm{id})\varphi_i=(\mathrm{id}\otimes\varphi_i)\varphi_i$$

$$(\varepsilon\otimes\mathrm{id})\varphi_i=\mathrm{id}$$

where Δ and ε are the coproduct and co-unit of $A(R)$, respectively. We now study the relationship between φ_is.

For REI, φ_{I} is defined by

$$\varphi_{\mathrm{I}}(K)=TKT^{-1}. \tag{13}$$

This means that if K is a solution to REI then $\varphi_{\mathrm{I}}(K)$ is also a solution to REI. From proposition 1 it follows that

$$(\varphi_{\mathrm{I}}(R))^{-1}=(TKT^{-1})^{-1}=TK^{-1}T^{-1}=TMT^{-1}$$

is a solution to REII. This suggest that φ_{II} is defined by

$$\varphi_{\mathrm{II}}(M)=(\varphi_{\mathrm{I}}(M^{-1})^{-1}. \tag{14}$$

If R satisfies the crossing symmetry, we can prove that [12]

$$T^tCT=TCT^t=C$$

where t stands for the matrix transposition. Let K be a solution of REI. Then

$$\begin{aligned}(\varphi_{\mathrm{I}}(K))^{-1}C^{-1}&=TK^{-1}T^{-1}C^{-1}=TK^{-1}(T^tCT)^{-1}T^t=TK^{-1}C^{-1}T^t\\ &=TNT^t=\varphi_{\mathrm{III}}(N)\end{aligned} \tag{15}$$

is a solution to REI. Therefore

$$\varphi_{\mathrm{III}}(N)=(\varphi_{\mathrm{I}}(C^{-1}N^{-1}))^{-1}C^{-1}. \tag{16}$$

The above discussions mean that if R satisfies the crossing symmetry and K (or M, N) is inversive then the co-module structure of $\mathscr{A}_i(R)$ is closely connected through a simple relation.

We consider the parameter-dependent form of reflection equations

$$R(xy^{-1})K_1(x)\tilde{R}(xy)K_2(y)=K_2(y)R(xy)K_1(x)\tilde{R}(xy^{-1}) \tag{17}$$

$$R(xy^{-1})M_1(y)R(xy)^{-1}M_2(x)=M_2(x)\tilde{R}(xy)^{-1}M_1(y)\tilde{R}(xy^{-1}) \tag{18}$$

$$R(xy^{-1})N_1(y)R(xy)^{t_1}N_2(x)=N_2(x)R(xy)^{t_1}N_1(y)R(xy^{-1}). \tag{19}$$

Introducing $\check{R}(x)=PR(x)$, equations (17) and (18) can be rewritten as

$$\check{R}(xy^{-1})K_1(x)\check{R}(xy)K_1(y)=K_1(y)\check{R}(xy)K_1(x)\check{R}(xy^{-1}) \tag{20}$$

$$\check{R}(xy^{-1})M_1(y)\check{R}(xy)^{-1}M_1(x)=M_1(x)\check{R}(xy)^{-1}M_1(y)\check{R}(xy^{-1}). \tag{21}$$

If we require that

$$\check{R}(x)\check{R}(x^{-1})=\rho(x)I$$

where $\rho(x)$ is an arbitrary function of x, then equation (20) becomes

$$\check{R}(xy^{-1})K_1(y)\check{R}(x^{-1}y^{-1})^{-1}K_1(x)=K_1(x)\check{R}(x^{-1}y^{-1})^{-1}M_1(y)\check{R}(xy^{-1}). \tag{22}$$

Changing $x\to y^{-1}$, $y\to x^{-1}$, we further have

$$\check{R}(xy^{-1})K_1(y^{-1})\check{R}(xy)^{-1}K_1(x^{-1})=K_1(x^{-1})\check{R}(xy)^{-1}M_1(y^{-1})\check{R}(xy^{-1}) \tag{23}$$

which is just equation (21) and

$$M(x)=K(x^{-1}). \tag{24}$$

Using the fact that the Yang–Baxterization procedure of the braid group representations $\check{R}$ preserves the crossing symmetry [13], namely

$$R(x)^{t_1}=F(x)C_1R(x_0x)^{-1}C_1^{-1} \tag{25}$$

where $F(x)$ is a function of x, and the relations

$$[PC_1C_2, R(x)]=0, \qquad [P, R(x)^{t_1}]=0$$

we can derive the parameter-dependent form of REIII:

$$\check{R}(xy^{-1})N(y)_1C_1\check{R}(x_0xy)^{-1}N(x)_1C_1=N(x)_1C_1\check{R}(x_0xy)^{-1}N_1C_1\check{R}(xy^{-1}). \tag{26}$$

Letting $x'=x_0^{1/2}x$, $y'=x_0^{1/2}y$, we find that

$$\begin{aligned}\check{R}(x'y'^{-1})N(x_0^{-1/2}y')_1C_1\check{R}(x'y')^{-1}N(x_0^{-1/2}x')_1C_1\\ =N(x_0^{-1/2}x')_1C_1\check{R}(x'y')^{-1}N(x_0^{-1/2}y')_1C_1\check{R}(x'y'^{-1}).\end{aligned} \tag{27}$$

Comparing (27) with (21) we find that

$$M(x)=N(x_0^{-1/2}x)C. \tag{28}$$

Now we present an explicit example. In [11] we obtained the Yang–Baxterization of REI for R given in (8), namely

$$K(x)=(x-x^{-1})K-\omega xI$$

then from (24) and (28) we obtain the Yang–Baterization as

$$M(x)=K(x^{-1})=-(x-x^{-1})K-\omega x^{-1}I$$

$$N(x)=-(q^{-1}x-qx^{-1})KC^{-1}-\omega q^{-1}x^{-1}C^{-1},$$

where $x_0=q^{-2}$ is used.

Here we have investigated the relationship of three different forms of reflection equations. Their solutions are connected through a simple transformation if R satisfies the crossing symmetry and K is inversive. In [6] the constant solutions to REI and REIII are studied. Indeed, their discussions are not included in this paper because the R-matrice related to the fundamental representations of quantum algebra $sl_q(n)$ do not satisfy the crossing symmetry except for $n=2$. The conclusions of this paper, however, would also be of significance for R associated with fundamental representations of quantum algebras $so_q(n)$ and $sp_q(n)$ satisfying the crossing symmetry because their constant solutions have not been constructed so far. According to our approach it is enough to construct constant solutions (with non-vanishing determinants) to one reflection solution.

Yang–Baxterization of REI for R with two distinct eigenvalues has been worked out in [11]. However, we have to consider Yang–Baxterization of RE for R with a cubic relation (three distinct eigenvalues) since the R related to fundamental representations of $so_q(n)$ and $sp_q(n)$ satisfy the cubic relation. If a Yang–Baxterization scheme of one reflection equation is established, then the other two reflection equations can be easily Yang–Baxterized using the procedure presented in this paper. We will consider these problems in a separate paper.

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References

[1] Cherednik I 1984 *Theor. Math. Phys.* **61** 55
[2] Reshetikhin N and Semenov-Tian-Shansky M 1990 *Lett. Math. Phys.* **19** 13
[3] Sklyanin E 1988 *J. Phys. A: Math. Gen.* **21** 2375
[4] Kulish P and Sklyanin E 1991 *J. Phys. A: Math. Gen.* **24** L435
[5] Kulish P and Sklyanin E 1992 *J. Phys. A: Math. Gen.* **25** 5963
[6] Kulish P, Saskai R and Schwiebert C 1992 *J. Math. Phys.* in press
[7] Fu H-C and Ge M-L 1993 *J. Phys. A: Math. Gen.* **26** L229
[8] Fu H-C and Ge M-L 1992 *J. Phys. A: Math. Gen.* **25** L1123
[9] Fu H-C and Ge M-L 1993 *Chinese Science Bulletin* **38** 497 (in Chinese)
[10] Fu H-C and Ge M-L 1993 q-boson realizations of reflection quadratic algebras associated with $GL(n)_q$ *Phys. Lett.* A in press
[11] Fu H-C, Ge M-L and Xue K 1993 *J. Phys. A: Math. Gen.* **26** L847
[12] Wadati M., Deguchi T. and Akutzu Y. 1989 *Phys. Rep.* **180** 247;
Takhtajan L 1990 *Introduction to Quantum Group and Integrable Massive Models of Quantum Field Theory* eds Ge M-L and Zhao B-H (Singapore: World Scientific)
[13] Cheng Y, Ge M-L and K Xue 1991 *Commun. Math. Phys.* **135** 486